Complex Numbers

2013

Q7 – 4 marks

Given that $z = 1-\sqrt{3}\,i$, write down $\bar{z}$ and express $\bar{z}^2$ in polar form. **4**

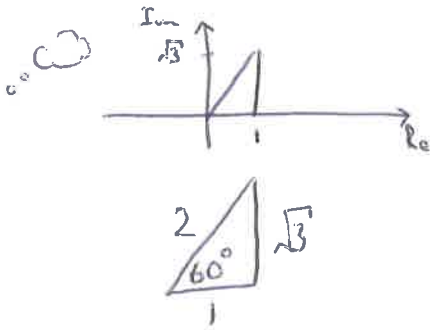
Written Solutions

$z = 1 - \sqrt{3}\,i$

$\bar{z} = 1 + \sqrt{3}\,i$

$\bar{z} = 2\left(\cos 60^\circ + i\sin 60^\circ\right)$

$\bar{z}^2 = 4\left(\cos 120^\circ + i\sin 120^\circ\right)$



or $\bar{z}^2 = 4\left(\cos \frac{2\pi}{3} + i\sin \frac{2\pi}{3}\right)$

2013

Q10 – 5 marks

Describe the loci in the complex plane given by:

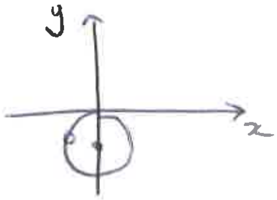
(*a*) $|z+i|=1$; 2

(*b*) $|z-1|=|z+5|$. 3

Written Solutions

z always 1 away from $-i$

$z = x+iy$



(a) $|z+i| = 1$

$|x+iy+i| = 1$

$|x+i(y+1)| = 1$

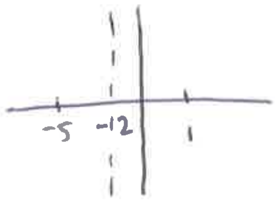
$\sqrt{x^2+(y+1)^2} = 1$

$x^2+(y+1)^2 = 1$

$\Rightarrow$ eqⁿ of a circle centre $(0,-1)$ radius 1

(b) $|z-1| = |z+5|$ $(\mathbb{R})$

the distance of z from the point 1 is the same as the distance ~~from~~ of point z from the point -5



$\Rightarrow |x+iy-1| = |x+iy+5|$

$\Rightarrow |(x-1)+iy| = |(x+5)+iy|$

$\Rightarrow \sqrt{(x-1)^2+y^2} = \sqrt{(x+5)^2+y^2}$

$\Rightarrow (x-1)^2+y^2 = (x+5)^2+y^2$

$\Rightarrow x^2-2x+1 = x^2+10x+25$

$\Rightarrow 12x = -24$

$\Rightarrow x = -2$

$\Rightarrow$ vertical line with eqⁿ $\underline{x=-2}$

2012

Q3 – 6 marks

Given that $(-1 + 2i)$ is a root of the equation

$$z^3 + 5z^2 + 11z + 15 = 0,$$

obtain all the roots. 4

Plot all the roots on an Argand diagram. 2

Written Solutions

$-1+2i$ is a root $\Rightarrow$ $-1-2i$ is also a root

quadratic factor $= z^2 - (-2)z + (1^2+2^2)$

$= \underline{\underline{z^2 + 2z + 5}}$

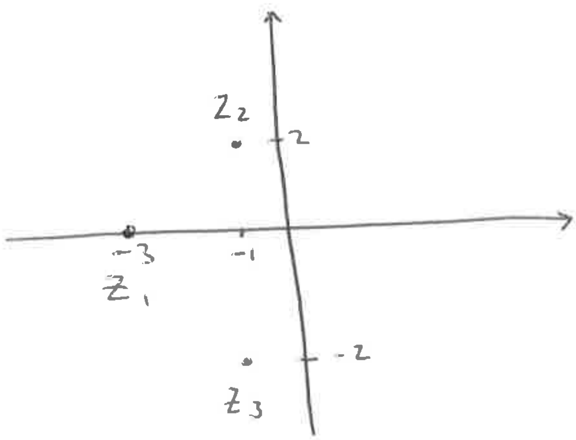
$(z+\alpha)(z^2+2z+5) = z^3+5z^2+11z+15$

$\Rightarrow \quad 5\alpha = 15$

$\alpha = 3$

$(z+3)(z^2+2z+5) = 0$

Roots are $\underline{\underline{z_1 = -3}}$, $\underline{\underline{z_2 = -1+2i}}$, $\underline{\underline{z_3 = -1-2i}}$



2012

Q16b – marks

(*a*) Prove by induction that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all integers $n \geq 1$. 6

(*b*) Show that the real part of $\dfrac{\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)^{11}}{\left(\cos\frac{\pi}{36}+i\sin\frac{\pi}{36}\right)^{4}}$ is zero. 4

Written Solutions

b) $\dfrac{\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)^{11}}{\left(\cos\frac{\pi}{36}+i\sin\frac{\pi}{36}\right)^{4}}$

$= \dfrac{\cos\frac{11\pi}{18}+i\sin\frac{11\pi}{18}}{\cos\frac{4\pi}{36}+i\sin\frac{4\pi}{36}}$

$= \cos\left(\frac{11\pi}{18}-\frac{\pi}{9}\right)+i\sin\left(\frac{11\pi}{18}-\frac{\pi}{9}\right)$

$= \cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$

$= 0 + 1i$

$= 0 + i$

$\Rightarrow$ real part $= 0$

2011

Q10 – 5 marks

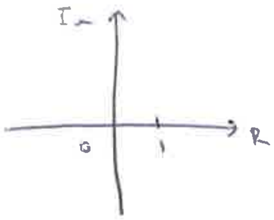
Identify the locus in the complex plane given by

$$|z-1| = 3.$$

Show in a diagram the region given by $|z-1| \le 3$. 5

Written Solutions

$|z-1| = 3$



or The distance between z and the number 1 is always 3

$\Rightarrow$ a circle of radius 3 around 1

Let $z = x + iy$

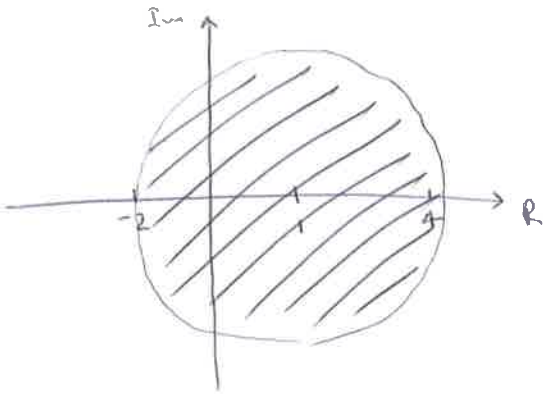
$z - 1 = (x-1) + iy$

So, $|(x-1) + iy| = 3$

$|(x-1) + iy|^2 = 9$

$(x-1)^2 + y^2 = 9$

$\Rightarrow$ circle centre $(1, 0)$ rad 3



2010

Q16 – 10 marks

Given $z = r(\cos\theta + i\sin\theta)$, use de Moivre's theorem to express z^3 in polar form. 1

Hence obtain $\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3$ in the form $a + ib$. 2

Hence, or otherwise, obtain the roots of the equation $z^3 = 8$ in Cartesian form. 4

Denoting the roots of $z^3 = 8$ by z_1, z_2, z_3:

(a) state the value $z_1 + z_2 + z_3$;

(b) obtain the value of $z_1^6 + z_2^6 + z_3^6$. 3

Written Solutions

$$z = r(\cos\theta + i\sin\theta)$$

$$z^3 = r^3(\cos\theta + i\sin\theta)^3$$

$$= r^3(\cos 3\theta + i\sin 3\theta)$$

$$\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3 = \cos 3\left(\frac{2\pi}{3}\right) + i\sin 3\left(\frac{2\pi}{3}\right)$$

$$= \cos 2\pi + i\sin 2\pi$$

$$= 1 + 0$$

in form $a + ib$

where $a = 1$, $b = 0$

"HENCE"

$$z^3 = 8 \qquad \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3 = 1$$

$$z^3 = 8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3$$

$$z = 2\left(\cos\left(\frac{2\pi}{3} + \frac{m2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3} + \frac{m2\pi}{3}\right)\right) \qquad m = 0, 1, 2$$

$$z_1 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$z_2 = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$z_3 = 2\left(\cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3}\right)$$

$$z_1 = 2\left(-\frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right) = -1 + \sqrt{3}\,i$$

$$z_2 = 2\left(-\frac{1}{2} - i\cdot\frac{\sqrt{3}}{2}\right) = -1 - \sqrt{3}\,i$$

$$z_3 = 2(1 + i0) = 2$$

PTO

"OTHERWISE" ①

$z^3 = 8 \rightarrow$ 

$z^3 = 8(\cos 0 + i\sin 0)$

$z = 8^{\frac{1}{3}}(\cos 0 + i\sin 0)^{\frac{1}{3}}$

$= 8^{\frac{1}{3}}\left(\cos\left(\frac{0+n2\pi}{3}\right) + i\sin\left(\frac{0+n2\pi}{3}\right)\right)$ $\quad n = 0, 1, 2$

$n=0 \quad z_1 = 2(\cos 0 + i\sin 0)$

$n=1 \quad z_2 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$n=2 \quad z_3 = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$z_1 = \underline{\underline{-1+\sqrt{3}i}}$

$z_2 = \underline{\underline{-1-\sqrt{3}i}}$

$z_3 = \underline{\underline{2}}$

"OTHERWISE" ②

$z^3 = 8$

$z^3 - 8 = 0$

$z^3 - 2^3 = 0$

$(z-2)(z^2+2z+4) = 0$

$z = 2$ or $z = \frac{-2 \pm \sqrt{-12}}{2}$ $\qquad \sqrt{12} = 2\sqrt{3}$

$\underline{\underline{z_1 = 2}}$, $z_2 = \underline{\underline{-1+4\sqrt{3}i}}$, $z_3 = \underline{\underline{-1-\sqrt{3}i}}$

a) $z_1 + z_2 + z_3 = \underline{\underline{0}}$

b) $z_1^6 + z_2^6 + z_3^6$ $\qquad z^3 = 8$

$= 8^2 + 8^2 + 8^2$ $\qquad z^6 = (z^3)^2$

$= \underline{\underline{192}}$

2009

Q6 – 6 marks

Express $z = \dfrac{(1+2i)^2}{7-i}$ in the form $a + ib$ where a and b are real numbers.

Show z on an Argand diagram and evaluate $|z|$ and $\arg(z)$. **6**

Written Solutions

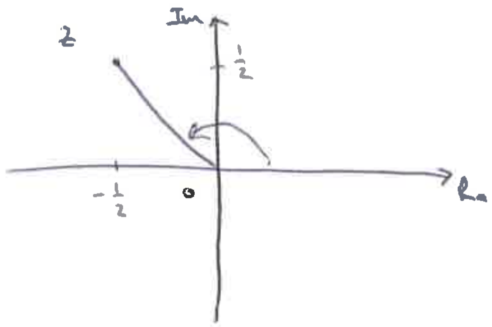
$$z = \frac{(1+2i)^2}{7-i}$$

$$= \frac{(1 + 4i - 4)(7+i)}{(7-i)(7+i)}$$

$$= \frac{-21 - 3i + 28i - 4}{49+1}$$

$$= \frac{-25 + 25i}{50}$$

$$= -\frac{1}{2} + \frac{1}{2}i$$



modulus is the length of the ray (line) that goes from the origin to the point itself

arg: is the angle round anti-clockwise from the ℝ axis unless it exceeds the ½ way mark in which case it is the other direction

So, $|z|^2 = \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

$$= \frac{1}{2}$$

$$|z| = \sqrt{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$

$$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{2}}{-\frac{1}{2}}\right)$$

$$= \tan^{-1}(-1)$$

$$= 135^\circ \text{ or } \frac{3\pi}{4}$$

2008

Q16 – 10 marks

Given $z = \cos\theta + i\sin\theta$, use de Moivre's theorem to write down an expression for z^k in terms of θ, where k is a positive integer.

Hence show that $\dfrac{1}{z^k} = \cos k\theta - i\sin k\theta$. **3**

Deduce expressions for $\cos k\theta$ and $\sin k\theta$ in terms of z. **2**

Show that $\cos^2\theta\ \sin^2\theta = -\dfrac{1}{16}\left(z^2 - \dfrac{1}{z^2}\right)^2$. **3**

Hence show that $\cos^2\theta\ \sin^2\theta = a + b\cos 4\theta$, for suitable constants a and b. **2**

Written Solutions

$$z^n = [\cos\theta + i\sin\theta]^n$$
$$= \cos(n\theta) + i\sin(n\theta)$$

$$\therefore\ z^k = [\cos\theta + i\sin\theta]^k$$
$$= \cos(k\theta) + i\sin(k\theta)$$

$$\frac{1}{z^k} = z^{-k} = [\cos\theta + i\sin\theta]^{-k}$$
$$= \cos(-k\theta) + i\sin(-k\theta)$$
$$= \cos(k\theta) - i\sin(k\theta)$$

$$z^k + \frac{1}{z^k} = \cos(k\theta) + i\sin(k\theta) + \cos(k\theta) - i\sin(k\theta)$$
$$= 2\cos(k\theta)$$

$$\therefore\ \cos(k\theta) = \frac{1}{2}\left[z^k + \frac{1}{z^k}\right]$$
$$= \underline{\underline{\frac{1}{2}[z^k + z^{-k}]}}$$

$$z^k - \frac{1}{z^k} = \cos(k\theta) + i\sin(k\theta) - [\cos(k\theta) - i\sin(k\theta)]$$
$$= 2i\sin(k\theta)$$

$$\therefore\ \sin(k\theta) = \underline{\underline{\frac{1}{2i}(z^k - z^{-k})}}$$

PTO

$\cos^2\theta \sin^2\theta$

$= (\cos\theta \sin\theta)^2$

$= \left[\frac{1}{2}(z^k + z^{-k}) \times \frac{1}{2i}(z^k - z^{-k})\right]^2$

$= \left[\frac{1}{4i}(z^k + z^{-k})(z^k - z^{-k})\right]^2$

$= \frac{1}{16i^2}(z^{2k} - z^{-2k})^2$

$= -\frac{1}{16}\left(z^{2k} - \frac{1}{z^{2k}}\right)^2$

$= -\frac{1}{16}\left(z^{4k} + \frac{1}{z^{4k}} - 2\right)$

$= -\frac{1}{16}(2\cos 4\theta - 2)$

$= -\frac{1}{8}\cos 4\theta + \frac{1}{8}$

$= \frac{1}{8} - \frac{1}{8}\cos 4\theta$

$\therefore \quad a = \frac{1}{8}$ and $b = -\frac{1}{8}$

2007

Q3 – 4 marks

Show that $z = 3 + 3i$ is a root of the equation $z^3 - 18z + 108 = 0$ and obtain the remaining roots of the equation. **4**

Written Solutions

$$z^3 - 18z + 108 = 0 \qquad z = 3 + 3i$$

$$(3+3i)^3 - 18(3+3i) + 108$$

$$= 3^3 + 3\cdot 3^2 \cdot (3i) + 3\cdot 3\cdot (3i)^2 + 1\cdot (3i)^3 - 54 - 54i + 108$$

$$= 27 + 81i - 81 - 27i - 54 - 54i + 108$$

$$= 0$$

$\Rightarrow$ $\underline{z_1 = 3+3i}$ is a root

$\Rightarrow$ $z_2 = 3-3i$ is also a root

$\Rightarrow$ ~~z_1~~ $z^2 - 6z + 18$ is a quadratic factor

$$z^3 - 18z + 108 = 0$$

$$\Rightarrow (z + 6)(z^2 - 6z + 18) = 0$$

if you have a quadratic factor, it is trivial to find the remaining linear factor

$\Rightarrow$ $\underline{z = -6}$, $\underline{z = 3+3i}$, $\underline{z = 3-3i}$

2007

Q11 – 4 marks

Given that $|z-2| = |z+i|$, where $z = x+iy$, show that $ax+by+c=0$ for suitable values of a, b and c. **3**

Indicate on an Argand diagram the locus of complex numbers z which satisfy $|z-2| = |z+i|$. **1**

Written Solutions

$|(x-2)+iy|^2 = |x+(y+1)i|^2$ $\qquad |z| = \sqrt{x^2+y^2}$

$(x-2)^2 + y^2 = x^2 + (y+1)^2$

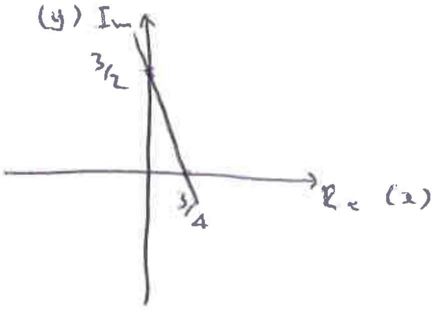
$x^2 - 4x + 4 + y^2 = x^2 + y^2 + 2y + 1$

$-4x - 2y + 3 = 0$

$\underline{\underline{4x + 2y - 3 = 0}}$ as req^d

$4x + 2y - 3 = 0$ "line" $\qquad$ loci: all the positions that can be occupied by pts

$\Rightarrow$ *



* $x = 0$, $y = 3/2$ $\qquad (0, 3/2)$

$y = 0$ $\quad x = 3/4$ $\qquad (3/4, 0)$